

# Co-existence of states in quantum systems

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**Abstract.** Co-existence of different states is a profound concept, which possibly underlies the phase transition and the symmetry breaking. Because of a property inherent to quantum mechanics (cf. uncertainty), the co-existence is expected to appear more naturally in quantum-microscopic systems than in macroscopic systems. In this paper a mathematical theory describing co-existence of states in quantum systems is presented, and the co-existence is classified into 9 types.

## 1. Introduction

The boundary-value problem of nonlinear partial differential equation of elliptic-type:

$$\begin{cases} -\nabla^2 u - mu - V(u)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

is studied, where  $m$  is a real number, and  $\Omega \in R^3$  is a closed domain with a sufficiently smooth boundary. The unknown complex function  $u$  consists of the unknown state  $\psi$  and the reference state  $\bar{\psi}$ :

$$u = \psi - \bar{\psi},$$

where  $\bar{\psi}$  (corresponding to a generalized concept of the vacuum) is not necessarily a solution of Eq. (1), although the most simplest case  $\bar{\psi} = 0$  (the simplest vacuum) satisfies Eq. (1). Let a part of the inhomogeneous term  $V(u)$ , whose spectral set is assumed to be included in a real axis, satisfy

$$\partial_u(V(u)u)|_{u=0} = V_L. \quad (2)$$

For the simplicity  $V_L$ , which corresponds to the signed strength of linearized interaction being independent of  $u$ , is assumed to be a real number. As is readily seen, the function  $u = \psi - \bar{\psi} = 0$  is always a solution of this problem (refer to the trivial solution). In this sense let us imagine a simple case when  $\bar{\psi} = 0$ , and then the emergence of a solution  $\psi$  from another solution  $\bar{\psi} = 0$  is true if  $u \neq 0$  is the solution of Eq. (1). Here we seek the non-trivial solution  $u \neq 0$  ( $\psi \neq \bar{\psi}$ ) to Eq. (1). The corresponding situation is nothing but the co-existence of different states  $\psi$  and  $\bar{\psi}$ .

Equation (1) is associated with the stationary problem of nonlinear Schrödinger equations as well as nonlinear Klein-Gordon equations. In the context of Klein-Gordon equations, it is possible to associate  $\sqrt{-m}$  with the mass (if  $m < 0$ ). Note that the statistical property inherent to many-body system, which might bring about rather interesting physical properties, is not taken into account in order to see the most fundamental properties associated with the co-existence in both nonlinear Schrödinger equations and nonlinear Klein-Gordon equations.

## 2. Theory describing the co-existence

### 2.1. Mathematical settings

Let  $X$  and  $Y$  be functional spaces

$$\begin{aligned} X &= W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega), \\ Y &= L^2(\Omega) \end{aligned}$$

respectively (for mathematical notation, see [1]). An inclusion relation  $X \subset Y$  is true. For  $u \in X$ , a mapping  $f : R^1 \times X \rightarrow Y$  is defined by

$$f(\lambda, u) := -\nabla^2 u - mu - V(u)u.$$

The original master equation is written by  $f(\lambda, u) = 0$ . Since the trivial solution  $u = 0$  always exists,  $f(\lambda, 0) = 0$  is satisfied. According to the Sobolev embedding theorem  $-\nabla^2$  is a  $C^2$ -mapping from  $X$  to  $Y$ , where the detail setting of  $V(u)$  is necessary to know the regularity of the mapping  $f$ . The space  $W_0^{1,2}(\Omega)$  denotes all the functions included in  $W^{1,2}(\Omega)$  satisfying  $u|_{\partial\Omega} = 0$ .

### 2.2. Linearized analysis

Linearized problem is derived. The Fréchet derivative of  $f(\lambda, u)$  is calculated as

$$f_u(\lambda, 0)[u] = -\nabla^2 u - \lambda u = 0, \quad (3)$$

where  $\lambda = m + V_L$ . This corresponds to the master equation for the linearized eigen-value problem. It is well known that the linearized problem (with the Dirichlet boundary condition) is solvable. Furthermore it is known that a infinite set of eigen-values  $\{\lambda_i\}_{i=0}^\infty$  of  $-\nabla^2$  satisfy

- $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ ;
- $\lambda_0$  is a simple eigen-value.

Let the eigen-function corresponding to the eigen-value  $\lambda_0$  be  $u_0$  (i.e.,  $-\nabla^2 u_0 = \lambda_0 u_0$ ). First, according to the simple property of the eigen-value  $\lambda_0$ , it is clear that

$$\text{Ker}(f_u(\lambda_0, 0)) = \{tu_0; t \in R^1\},$$

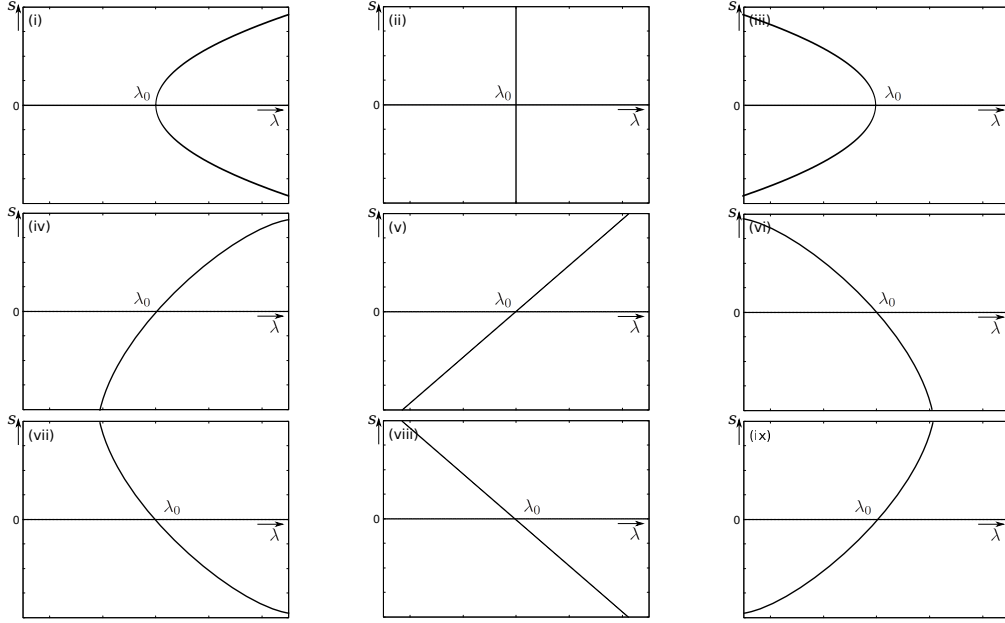
so that the dimension of  $\text{Ker}(f_u(\lambda_0, 0))$  is equal to 1. Second, if there exists a solution  $v \in X$  for  $\nabla^2 v - \lambda_0 v = h$  with  $h \in Y$ , then

$$R(f_u(\lambda_0, 0)) = \left\{ h \in Y; \int_{\Omega} h(x)u(x)dx = 0 \right\},$$

so that  $R(f_u(\lambda_0, 0))$  is a closed subset of  $Y$  with its co-dimension 1 (cf. the Riesz-Schauder theory [1]). Third, it is valid that

$$f_{u\lambda}(\lambda_0, 0)[u] = -\lambda_0 u \notin R(f_u(\lambda_0, 0)). \quad (4)$$

Consequently, according to the bifurcation theory [2, 3],  $\lambda = \lambda_0$  has been clarified to be a bifurcation point (corresponding to  $(\lambda_0, 0)$  in Fig. 1). Note that only sufficient conditions for the existence of the bifurcation point is presented in the bifurcation theory.



**Figure 1.** 9 types of co-existence based on Eqs. (5) and (6): cases (i), (ii), and (iii) appear if  $\mu_s(0) = 0$ , cases (iv), (v), and (vi) appear if  $\mu_s(0) > 0$ , cases (vii), (viii), and (ix) appear if  $\mu_s(0) < 0$ ; cases (i), (iv), and (vii) appear if  $\mu_{ss}(0) > 0$ , cases (ii), (v), and (viii) appear if  $\mu_{ss} = 0$ , cases (iii), (vi), and (ix) appear if  $\mu_{ss} < 0$ .

### 2.3. Nonlinear analysis

Co-existence of different states (i.e., existence of non-trivial solution  $u \neq 0$ ) is shown. We set a closed interval  $[-\epsilon_0, \epsilon_0]$  and a  $C^1$ -function  $\lambda(s)$  satisfying  $\lambda(0) = \lambda_0$ , where  $s$  parametrizes the functional space  $X$ . Under the three conditions confirmed in Sec. 2.2, let the corresponding solution  $u$  be represented by

$$u(\lambda, s, x) = su_0(\lambda, x) + sz(\lambda, s, x),$$

where  $s$  is defined on the interval, and  $z(\lambda, s, x)$  is a sufficiently smooth function of  $s$  defined on  $R^1 \times R^1 \times X$ . The function  $z(\lambda, s, x)$  satisfies  $z(\lambda, 0, x) = 0$  and

$$\int_{\Omega} z(x)u_0(x)dx = 0.$$

The function  $u(\lambda, s, x)$  satisfies the condition  $u(\lambda, 0, x) = 0$ , which means the existence of the trivial solution. It is useful to define a linear operator

$$A := -\nabla^2 - \lambda_0,$$

with its domain  $X$ , and then it is readily seen that  $A$  is a self-adjoint operator in  $Y$ . The original equation is written by  $Au = \mu(s)u + (V(u) - V_L)u$  with  $\mu(s) = \lambda(s) - \lambda_0$ , and the linearized problem is written by  $Au_0 = 0$ . By differentiating the original equation with respect to  $s$ , step

by step

$$\begin{aligned}
(Au)_s &= \mu_s u + \mu u_s + \partial_s(V(u)u) - V_L u_s \\
&= \mu_s u + \mu u_s + (\partial_s V(u))u + V(u)u_s - V_L u_s \\
(Au)_{ss} &= \mu_{ss} u + 2\mu_s u_s + \mu u_{ss} + \partial_s^2(V(u)u) - V_L u_{ss} \\
&= \mu_{ss} u + 2\mu_s u_s + \mu u_{ss} + (\partial_s^2 V(u))u + 2(\partial_s V(u))u_s + V(u)u_{ss} - V_L u_{ss} \\
(Au)_{sss} &= \mu_{sss} u + 3\mu_{ss} u_s + 3\mu_s u_{ss} + \mu u_{sss} + \partial_s^3(V(u)u) - V_L u_{sss} \\
&= \mu_{sss} u + 3\mu_{ss} u_s + 3\mu_s u_{ss} + \mu u_{sss} + (\partial_s^3 V(u))u + 3(\partial_s^2 V(u))u_s + 3(\partial_s V(u))u_{ss} \\
&\quad + V(u)u_{sss} - V_L u_{sss}
\end{aligned}$$

where the functions are represented by  $u_s = u_0 + sz_s + z$ ,  $u_{ss} = sz_{ss} + 2z_s$ , and  $u_{sss} = sz_{sss} + 3z_{ss}$  respectively. The derivatives of the inhomogeneous terms become

$$\begin{aligned}
\partial_s V(u) &= (\partial_u V(u)) u_s \\
\partial_s^2 V(u) &= (\partial_u^2 V(u)) u_s^2 + (\partial_u V(u)) u_{ss} \\
\partial_s^3 V(u) &= (\partial_u^3 V(u)) u_s^3 + 3(\partial_u^2 V(u)) u_s u_{ss} + (\partial_u V(u)) u_{sss}.
\end{aligned}$$

By taking  $s = 0$ , the bi-linear forms become

$$\begin{aligned}
(Au)_s|_{s=0} &= \mu_s(0)u|_{s=0} + \mu(0)u_s|_{s=0} + \partial_u(V(u)u)u_s|_{s=0} - V_L u_s|_{s=0} \\
&= V_L u_s|_{s=0} - V_L u_s|_{s=0} = 0, \\
((Au)_s|_{s=0}, u_0) &= 0,
\end{aligned}$$

$$\begin{aligned}
(Au)_{ss}|_{s=0} &= \mu_{ss}(0)u|_{s=0} + 2\mu_s(0)(u_0 + z|_{s=0}) + 2\mu(0)z_s|_{s=0} + \partial_s^2(V(u)u)|_{s=0} - 2V_L z_s|_{s=0} \\
&= 2\mu_s(0)u_0 + \partial_s^2(V(u)u)|_{s=0} - 2V_L z_s|_{s=0}, \\
((Au)_{ss}|_{s=0}, u_0) &= 2\mu_s(0)(u_0, u_0) + (\partial_s^2(V(u)u)|_{s=0}, u_0) - (2V_L z_s|_{s=0}, u_0),
\end{aligned}$$

$$\begin{aligned}
(Au)_{sss}|_{s=0} &= \mu_{sss}(0)u|_{s=0} + 3\mu_{ss}(0)(u_0 + z|_{s=0} + 6\mu_s(0)z_s|_{s=0} + 3\mu(0)z_{ss}|_{s=0} \\
&\quad + \partial_s^3(V(u)u)|_{s=0} - 3V_L z_{ss}|_{s=0} \\
&= 3\mu_{ss}(0)u_0 + 6\mu_s(0)z_s|_{s=0} + \partial_s^3(V(u)u)|_{s=0} - 3V_L z_{ss}|_{s=0}, \\
((Au)_{sss}|_{s=0}, u_0) &= 3\mu_{ss}(0)(u_0, u_0) + (6\mu_s(0)z_s|_{s=0}, u_0) + (\partial_s^3(V(u)u)|_{s=0}, u_0) - 3(V_L z_{ss}|_{s=0}, u_0),
\end{aligned}$$

where  $u|_{s=0} = 0$ ,  $z|_{s=0} = 0$ , and  $\mu(0) = 0$  are utilized, as well as Eq. (2).  $(Au_{ss}|_{s=0}, u_0) = (u_{ss}|_{s=0}, Au_0) = 0$  due to  $Au_0 = 0$ . Consequently

$$2\mu_s(0) = -(\partial_s^2(V(u)u)|_{s=0}, u_0) + 2(V_L z_s|_{s=0}, u_0), \quad (5)$$

and the sign of  $\lambda_s(0) = \mu_s(0)$  is determined by  $-(\partial_s^2(V(u)u)|_{s=0}, u_0) + 2(V_L z_s|_{s=0}, u_0)$ . In the same manner  $(Au_{sss}|_{s=0}, u_0) = (u_{sss}|_{s=0}, Au_0) = 0$ . It leads to

$$3\mu_{ss}(0) = -(\partial_s^3(V(u)u)|_{s=0}, u_0) - (6\mu_s(0)z_s|_{s=0}, u_0) + 3(V_L z_{ss}|_{s=0}, u_0), \quad (6)$$

and the sign of  $\lambda_{ss}(0) = \mu_{ss}(0)$  is determined by  $-(\partial_s^3(V(u)u)|_{s=0}, u_0) - (6\mu_s(0)z_s|_{s=0}, u_0) + 3(V_L z_{ss}|_{s=0}, u_0)$ . In particular, if  $\lambda_s(0) = \mu_s(0) = 0$  is true, the sign of  $\lambda_{ss}(0)$  is determined by  $-(\partial_s^3(V(u)u)|_{s=0}, u_0) + 3(V_L z_{ss}|_{s=0}, u_0)$ . According to Eqs. (5) and (6), the co-existence of states is classified into 9 types (Fig. 1). In Figure 1, around the neighbour of the bifurcation point  $(\lambda_0, 0)$ , two solutions co-exist in types (iv) to (ix), while the transition from single-existence to co-existence is described in types (i) and (iii).

**Table 1.** Systematic analysis for  $\psi^k$ -interaction theory. Possible classification of co-existence is shown in the column “Type”, where  $\sigma = 4\eta(u_0 z_s|_{s=0}, u_0) - 2\eta(u_0^2, u_0)(z_s|_{s=0}, u_0)$ .

$k$	$\partial_s V(u)$	$\partial_s^2 V(u)$	$\partial_s^2(V(u)u) _{s=0}$	$\partial_s^3(V(u)u) _{s=0}$	$\mu_s(0)$	$\mu_{ss}(0)$	Type
$= 3$	$-\eta u_s$	$-\eta u_{ss}$	$-2\eta u_0^2$	$-12\eta u_0 z_s _{s=0}$	$\eta(u_0^2, u_0)$	$\sigma$	all
$= 4$	$0$	$-2\eta u_s^2$	$0$	$-6\eta u_0^3$	$0$	$2\eta(u_0^3, u_0)$	(i), (ii), (iii)
$\geq 5$	$0$	$0$	$0$	$0$	$0$	$0$	(ii)

### 3. Application to $\psi^k$ -interaction theory

If the Lagrangian includes the  $k$ th-order nonlinearity in its interaction part (for example, see textbooks of particle physics), the inhomogeneous term of the master equation becomes

$$V(u)u = -\eta u^{k-1},$$

for integers  $k \geq 1$ , where  $\eta$  is assumed to be a real number. Here  $V_L = 0$  and  $V(u)|_{s=0} = V(u)|_{u=0} = 0$  are true. The first derivative is

$$\partial_u V(u)|_{u=0} = -(k-2)\eta u^{k-3}|_{u=0}$$

for  $k \geq 3$ , so that it is equal to  $-\eta$  for  $k = 3$ , and zero for  $k \geq 4$ . The second derivative is

$$\partial_u^2 V(u)|_{u=0} = -(k-2)(k-3)\eta u^{k-4}|_{u=0}$$

for  $k \geq 4$ , so that it is equal to zero for  $k = 3$ ,  $-2\eta$  for  $k = 4$ , and zero for  $k \geq 5$ . The third derivative is

$$\partial_u^3 V(u)|_{u=0} = -(k-2)(k-3)(k-4)\eta u^{k-5}|_{u=0}$$

for  $k \geq 5$ , so that it is equal to zero for  $k \leq 4$ ,  $-6\eta$  for  $k = 5$ , and zero for  $k \geq 6$ . Results are summarized in Table 1. In case of  $k = 4$  ( $\psi^4$ -interaction theory), the non-trivial solution corresponds to type (i) of Fig. 1 if  $\eta > 0$ , to type (ii) if  $\eta = 0$ , and to type (iii) if  $\eta < 0$ . In particular when  $\eta > 0$ , the co-existence emerges only if  $m > \lambda_0$  (cf. spontaneous symmetry breaking).

If there is no interaction (free particle condition:  $\eta = 0$ ),  $\mu_s(0) = \mu_{ss}(0) = 0$  is true, and the co-existence is classified into type (ii). If the interaction is linear ( $V(u) = V_L \neq 0$ ;  $\psi^2$ -interaction theory), the derivatives are

$$\partial_u V(u)|_{u=0} = \partial_u^2 V(u)|_{u=0} = \partial_u^3 V(u)|_{u=0} = 0,$$

so that  $\partial_s V(u) = \partial_s^2 V(u) = \partial_s^3 V(u) = 0$ . It leads to  $\partial_s^2(V(u)u)|_{s=0} = V_L u_{ss}|_{s=0}$  and  $\partial_s^3(V(u)u)|_{s=0} = V_L u_{sss}|_{s=0}$  so that  $\mu_s(0) = -(V_L z_s|_{s=0}, u_0) + (V_L z_s|_{s=0}, u_0) = 0$  and  $\mu_{ss}(0) = -(V_L z_{ss}|_{s=0}, u_0) + (V_L z_{ss}|_{s=0}, u_0) = 0$  follows. The co-existence is classified into type (ii). As a result the nonlinearity can be identified by the classification other than type (ii).

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